

Bäcklund transformations between four Lax-integrable 3D equations

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Abstract. We show that four Lax-integrable 3D differential equations are related via Bäcklund transformations.

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The aim of this note is to construct Bäcklund transformations, [7, 4], between the following four equations

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \quad (1)$$

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \quad (2)$$

$$u_{yy} = u_y u_{tx} - u_x u_{ty}, \quad (3)$$

$$u_{ty} = u_t u_{xy} - u_y u_{tx}, \quad (4)$$

In different contexts, these equations have been appeared before in [3, 15, 5, 9, 16, 1, 8, 13, 14, 10, 11, 12, 6, 2]. The equations are Lax-integrable, that is, each equation has a differential covering (or a Lax pair) with a non-removable spectral parameter. The coverings for (1), (2), (3), (4), are defined, [15, 5, 16, 10, 9, 1], by the following over-determined systems, respectively:

$$\begin{cases} v_t &= (\lambda^2 - \lambda u_x - u_y) v_x, \\ v_y &= (\lambda - u_x) v_x, \end{cases} \quad (5)$$

$$\begin{cases} v_t &= (u_x - \lambda) v_x, \\ v_y &= \lambda^{-1} u_y v_x, \end{cases} \quad (6)$$

$$\begin{cases} v_t &= \lambda^{-1} u_y^{-1} v_y, \\ v_x &= (\lambda + u_y u_x^{-1}) v_y, \end{cases} \quad (7)$$

$$\begin{cases} v_t &= \lambda (\lambda + 1)^{-1} u_t v_x, \\ v_y &= \lambda u_y v_x. \end{cases} \quad (8)$$

Parameter $\lambda \in \mathbb{R}$ is assumed to satisfy conditions $\lambda \neq 0$ in (6), (7) and $\lambda \notin \{-1, 0\}$ in (8). The compatibility conditions for (5), (6), (7), and (8) coincide with equations (1), (2), (3), and (4), respectively. Excluding u from (5), (6), (7) with $\lambda = 1$ and from (8) for arbitrary λ yields equations

$$v_{yy} = v_{tx} + \frac{v_y - v_t}{v_x} v_{xx} + \frac{v_y - v_x}{v_x} v_{xy}, \quad (9)$$

$$v_{ty} = \frac{v_t + v_x}{v_x} v_{xy} - \frac{v_y}{v_x} v_{xx}, \quad (10)$$

$$v_{yy} = \frac{v_y}{v_t} v_{tx} + \frac{v_y - v_x}{v_t} v_{ty}, \quad (11)$$

$$v_{ty} = \frac{\lambda + 1}{\lambda} \frac{v_t}{v_x} v_{xy} - \frac{1}{\lambda} \frac{v_y}{v_x} v_{tx}. \quad (12)$$

In other words, systems (5), (6), (7), (8) define Bäcklund transformations between pairs of equations (1) and (9), (2) and (10), (3) and (11), (4) and (12), respectively. Equation (12) was considered in [17].

THEOREM. *The following pairs of equations are equivalent via point transformations:*

- (i) (9) and (2),
- (ii) (10) and (11),
- (iii) (11) and (4).

Proof. (i) Write equation (9) as

$$\tilde{v}_{\tilde{y}\tilde{y}} = \tilde{v}_{\tilde{t}\tilde{x}} + \frac{\tilde{v}_{\tilde{y}} - \tilde{v}_{\tilde{t}}}{\tilde{v}_{\tilde{x}}} \tilde{v}_{\tilde{x}\tilde{x}} + \frac{\tilde{v}_{\tilde{y}} - \tilde{v}_{\tilde{x}}}{\tilde{v}_{\tilde{x}}} \tilde{v}_{\tilde{x}\tilde{y}}. \quad (13)$$

Then the change of variables

$$\tilde{t} = t, \quad \tilde{x} = -u + x, \quad \tilde{y} = x, \quad \tilde{v} = y \quad (14)$$

maps equation (13) to equation (2).

(ii) Write equation (11) as

$$\tilde{v}_{\tilde{y}\tilde{y}} = \frac{\tilde{v}_{\tilde{y}}}{\tilde{v}_{\tilde{t}}} \tilde{v}_{\tilde{t}\tilde{x}} + \frac{\tilde{v}_{\tilde{y}} - \tilde{v}_{\tilde{x}}}{\tilde{v}_{\tilde{t}}} \tilde{v}_{\tilde{t}\tilde{y}}. \quad (15)$$

Then the change of variables

$$\tilde{t} = y, \quad \tilde{x} = t, \quad \tilde{y} = -x, \quad \tilde{v} = v \quad (16)$$

maps equation (15) to equation (10).

(iii) The change of variables

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{y} = u, \quad \tilde{v} = y \quad (17)$$

maps equation (15) to equation (4). \square

COROLLARY. *Each pair of equations (1), (2), (3), (4), (9), (10), (11), (12) is related via an appropriate combination of transformations (5), (6), (7), (8), (14), (16), (17).*

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